Two Results on Separating Vectors

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Abstract. We show that a linear space of bounded operators with denumerable Hamel basis has a separating vector if and only if its subspace of finite rank operators has a separating vector, and also that the weakly closed algebra generated by a triangular operator has a separating vector. Moreover, we prove density of the set of all separating vectors.

A *separating vector* for a linear subspace $\mathscr{S}$ of linear transformations on a vector space V is a vector $x \in V$ for which the map $S \to Sx$ from $\mathscr{S}$ to V is injective. The purpose of this note is to settle two separating vector problems which arose in our earlier work and which have been open for several years.

In Theorem 8 we answer a question in [L], which arose in a study of reflexivity, and show that if $\mathscr{S}$ is a linear subspace of $\mathscr{B}(X)$ for a Banach space X, and if $\mathscr{S}$ has a denumerable Hamel basis, then $\mathscr{S}$ has a separating vector if and only if the linear subspace $\mathscr{S}_F$ of finite rank operators in $\mathscr{S}$ has a separating vector. Moreover, in this case $\operatorname{sep}(\mathscr{S})$ is dense in X, where $\operatorname{sep}(\mathscr{S})$ denotes the set of *all* separating vectors for $\mathscr{S}$. In the types of applications we have in mind, X will often be an operator algebra.

In Corollary 12 we show that if T is a Hilbert space operator which is triangular in the sense that there is an orthonormal basis $\{e_i\}$, such that $Te_n \in \operatorname{span}\{e_1, \dots, e_n\}$ for each n, then the weak operator topology closed algebra $\mathscr{W}(T)$ generated by T and I has a separating vector. (In fact, we prove the stronger result that $\operatorname{sep}(\mathscr{W}(T))$ is dense, and we obtain this in a somewhat wider setting.) The question of validity of this was raised formally in [LW1] and [HLW], although it had been considered earlier, and was motivated by a counterexample constructed in [W] of a Hilbert space operator S for which $\mathscr{W}(S)$ fails to have a separating vector. Theorems 8 and 11 are related in that the ideas that led to a proof of the first inspired the proof of the second.

We refer the reader to [H] for an extensive exposition of triangular operators, and to [LW1] for a discussion of some open questions. In this article all vector spaces will be assumed complex. If V is a vector space we use $\mathscr{L}(V)$ to denote

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all linear transformations on V, and if X is a Banach space, we use $\mathscr{B}(X)$ to denote all *bounded* linear transformations on X. By a *complex line* we mean a one dimensional affine subspace.

We prove a sequence of results leading to Theorem 8. Some have independent interest.

Lemma 1. *Let $\mathscr{S}$ be a linear subspace of $\mathscr{L}(V)$ and let $x \in \operatorname{sep}(\mathscr{S})$. Let $\{S_1, \dots, S_n\}$ be a finite set of nonzero elements of $\mathscr{S}$. If there is a vector $y \in V$ and distinct scalars $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that*

$$S_i(y + \lambda_i x) = 0, \qquad i = 1, \dots, n,$$

then $\{S_1, \dots, S_n\}$ is linearly independent.

Proof. Suppose $c_1 S_1 + \cdots + c_n S_n = 0$ for scalars $c_i \in \mathbb{C}$. We will show $c_i = 0$, $1 \le i \le n$. The hypothesis implies that

$$c_1 S_1(y + \lambda_1 x) + \cdots + c_n S_n(y + \lambda_n x) = 0.$$

Rewrite this as

$$(c_1 S_1 + \cdots + c_n S_n)y + (c_1 \lambda_1 S_1 + \cdots + c_n \lambda_n S_n)x = 0.$$

The first term vanishes by hypothesis so, since x separates $\mathscr{S}$, this implies that

$$c_1 \lambda_1 S_1 + \cdots + c_n \lambda_n S_n = 0.$$

Repeating the argument above with $c_i \lambda_i$ in place of c_i, we obtain

$$c_1 \lambda_1^2 S_1 + \cdots + c_n \lambda_n^2 S_n = 0.$$

Proceeding inductively, this implies that

$$c_1 \lambda_1^\ell S_1 + \cdots + c_n \lambda_n^\ell S_n = 0$$

for all positive integers ℓ. It follows that

$$c_1 p(\lambda_1) S_1 + c_2 p(\lambda_2) S_1 + \cdots + c_n p(\lambda_n) S_n = 0$$

for all complex polynomials $p(z)$. Fix i and choose $p(z)$ so that $p(\lambda_i) = 1$ and $p(\lambda_j) = 0$ for $j \ne i$. This shows that $c_i S_i = 0$, and hence $c_i = 0$ since $S_i \ne 0$. $\square$

Lemma 2. *Let $\mathscr{S}$ be a linear subspace of $\mathscr{L}(V)$ of finite dimension n. Let $y \in V$, and suppose $x \in \operatorname{sep}(\mathscr{S})$. If there exist distinct nonzero scalars $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that $y + \lambda_i x$ fails to separate $\mathscr{S}$ for each $1 \le i \le n$, then $y \in \operatorname{sep}(\mathscr{S})$.*

Proof. By hypothesis there are nonzero elements $S_1, \dots, S_n$ of $\mathscr{S}$ such that

$$S_i(y + \lambda_i x) = 0$$

for each $i = 1, \dots, n$. Then $\{S_1, \dots, S_n\}$ are linearly independent by Lemma 1, so span the space $\mathscr{S}$ since $\mathscr{S}$ has dimension n. If $(c_1S_1 + \cdots + c_nS_n)\ y = 0$ for some $c_i \in \mathbb{C}$, it follows from the equation

$$c_1S_1(y + \lambda_1 x) + \cdots + c_nS_n(y + \lambda_n x_n) = 0$$

that $(c_1\lambda_1S_1 + \cdots + c_n\lambda_nS_n)x = 0$. By hypothesis x separates $\mathscr{S}$, so this implies that $c_1\lambda_1S_1 + \cdots + c_n\lambda_nS_n = 0$ and hence $c_i\lambda_i = 0$, $1 \le i \le n$. Since the λ_i are nonzero, this shows that $c_i = 0$, $1 \le i \le n$. Since $(c_1S_1 + \cdots + c_nS_n)$ represents a generic element of $\mathscr{S}$, this shows that y separates $\mathscr{S}$.

Proposition 3. *Let $\mathscr{S}$ be a linear subspace of $\mathscr{L}(V)$ of finite dimension n. If L is a complex line in V, then either L contains no separating vector for $\mathscr{S}$ or all but at most n vectors in L separate $\mathscr{S}$.*

Proof. Suppose $x \in L$ separates $\mathscr{S}$. By way of contradiction, suppose $x_1, \dots, x_{n+1}$ are distinct elements of L which fail to separate $\mathscr{S}$. Let $y = x_{n+1}$. Since $y \ne x$ and both lie in L, we have

$$L = \{(1-\alpha)y + \alpha x \colon\ \alpha \in \mathbb{C}\}.$$

For each $1 \le i \le n$ let α_i be the unique scalar with $x_i = (1-\alpha_i)y + \alpha_i x$. For each i we have $\alpha_i \ne 1$ since $x_i \ne x$. Let $\lambda_i = \alpha_i(1-\alpha_i)^{-1}$, $1 \le i \le n$. The λ_i are distinct because the α_i are distinct. We have $y + \lambda_i x = (1-\alpha_i)^{-1}x_i$ so $y + \lambda_i x$ fails to separate $\mathscr{S}$, $1 \le i \le n$. An application of Lemma 2 now implies that y must separate $\mathscr{S}$, our contradiction. $\square$

Lemma 2 and Proposition 3 have generalizations to subspaces $\mathscr{S}$ of algebraic dimension $\aleph_0$. No topological assumptions are needed for this, as opposed to the case of some of the results in [L].

Proposition 4. *Let $\mathscr{S}$ be a linear subspace of $\mathscr{L}(V)$ of denumerable Hamel basis. Then:*

(i) *If x separates $\mathscr{S}$ and $y \in V$ is arbitrary, there is at most countably many complex numbers λ for which $y + \lambda x$ fails to separate $\mathscr{S}$.*
(ii) *If L is a complex line in V, then either L contains no separating vectors for $\mathscr{S}$ or all but countably many vectors in L separates $\mathscr{S}$.*

Proof. Write $\mathscr{S} = \operatorname{span}\{S_n\}_{n=1}^{\infty}$ and let $\mathscr{S}_n = \operatorname{span}\{S_1, \dots, S_n\}$. If $x \in V$, then x separates $\mathscr{S}$ iff x separates $\mathscr{S}_n$ for all n. So (ii) follows from Proposition 3. Then (i) follows from the fact that for $\lambda \neq -1$, $y + \lambda x = (1-\alpha)^{-1}[(1-\alpha)y + \alpha x]$ for $\alpha = \lambda(1+\lambda)^{-1}$. □

Corollary 5. *Let $\mathscr{S}$ be a linear subspace of continuous linear transformations on a topological vector space X, and suppose $\mathscr{S}$ has a denumerable Hamel basis. Then* $\operatorname{sep}(\mathscr{S})$ *is either empty or dense in X.*

Proof. If $x \in \operatorname{sep}(\mathscr{S})$ and $y \in X$ is arbitrary, let L be the complex line containing x and y. By Proposition 4, $\operatorname{sep}(\mathscr{S}) \cap L$ is dense in L so its closure contains y. □

Proposition 6. *Let X be a Banach space and let $\mathscr{S}$ be a linear subspace of $\mathscr{B}(X)$ of finite dimension. Then sep($\mathscr{S}$) is an open subset of X.*

Proof. We will show that $X \setminus \operatorname{sep}(\mathscr{S})$ is closed. Suppose $x_n \in X \setminus \operatorname{sep}(\mathscr{S})$ and $x_n \to x$. For each n there exists $S_\ell \in \mathscr{S}$ with $\|S_\ell\| = 1$ such that $S_\ell x_\ell = 0$. By compactness there exists a subsequence $\{S_{n_\ell}\}_{\ell=1}^{\infty}$ which converges to a nonzero operator $S \in \mathscr{S}$. We have

$$Sx = \lim_{\ell} S_{n_\ell} x_{n_\ell} = 0.$$

So $x \in X \setminus \operatorname{sep}(\mathscr{S})$. □

Lemma 7. *Let X be a Banach space and let $\mathscr{S}_n \subseteq \mathscr{S}_{n+1}$ be an increasing sequence of finite dimensional subspaces. If each $\mathscr{S}_n$ has a separating vector, then $S := \bigcup_{n=1}^{\infty} \mathscr{S}_n$ has a separating vector.*

Proof. We have $\operatorname{sep}(\mathscr{S}) = \bigcap_{n=1}^{\infty} \operatorname{sep}(\mathscr{S}_n)$. By Corollary 5 and Proposition 6, $\operatorname{sep}(\mathscr{S}_n)$ is open and dense in X. So, since X is complete, the Baire Category Theorem implies that $\bigcap_{n=1}^{\infty} \operatorname{sep}(\mathscr{S}_n)$ is nonempty. □

Theorem 8. *Let X be a Banach space and let $\mathscr{S}$ be a linear subspace of $\mathscr{B}(X)$ with denumerable Hamel basis. Then $\mathscr{S}$ has a separating vector if and only if $\mathscr{S}_F$ has a separating vector.*

Proof. Only one direction requires proof. Assume $\mathscr{S}_F$ has a separating vector. By [L, Proposition 2.3] the theorem is true if $\mathscr{S}$ has finite dimension. If $\mathscr{S}$ has infinite dimension, let $\{S_n\}_{n=1}^{\infty}$ be a countable spanning set for $\mathscr{S}$ and let $\mathscr{S}_n = \operatorname{span}\{S_1, \dots, S_n\}$, $1 \le n < \infty$. Our hypothesis implies that $(\mathscr{S}_n)_F$ has a separating vector and thus, by the finite dimensional result, that $\mathscr{S}_n$ has a separating vector. Then $\mathscr{S}$ has a separating vector by Lemma 7. □

We can extend the notion of separating vector in the following fashion: If $\mathscr{S} \subseteq \mathscr{L}(V)$, define a (right) *separator* for $\mathscr{S}$ to be an *operator* T in $\mathscr{L}(V)$ for which the map $S \to ST$ is injective. Define *left* separator accordingly. (Then $\mathscr{S}$ has a right separator iff $\mathscr{S}^*$ has a left separator.)

Proposition 9. *The following are equivalent:*

(i) *$\mathscr{S}$ has a right separator of rank n.*
(ii) *There is an n-tuple $(u_1, \dots, u_n)$ of vectors in V such that no nonzero element of $\mathscr{S}$ annihilates every u_i (a separating n-tuple).*
(iii) *The n-fold inflation $\mathscr{S}^{(n)} := \{S \oplus \cdots \oplus S \colon S \in \mathscr{S}\}$ acting on $V^{(n)} = V \oplus \cdots \oplus V$ has a separating vector.*

Proof. If T is a right separator, then any n-tuple $(u_1, \dots, u_n)$ which spans $\mathrm{ran}(T)$ separates $\mathscr{S}$, and conversely, given $(u_1, \dots, u_n)$ any T whose range contains $\{u_1, \dots, u_n\}$ will do. It is clear that $(u_1, \dots, u_n)$ separates $\mathscr{S}$ iff the vector $u_1 \oplus \cdots \oplus u_n$ separates $\mathscr{S}^{(n)}$. □

Remark 10. Using Proposition 9, most of the previous results easily generalize to separators and separating n-tuples, and relate to inflations. The other results easily adapt using the observation that T is a right separator for $\mathscr{S}$ iff T is a separating *vector* for the left regular representation of $\mathscr{S}$ on $\mathscr{L}(V)$. The statements of Lemmas 1 and 2 and Propositions 3 and 4 are valid with "separating vector $x \in V$" replaced with "right separator $x \in \mathscr{L}(V)$" and "vector $y \in V$" replaced with "operator $y \in \mathscr{L}(V)$". Here "complex line in V" should be replaced with "complex line in $\mathscr{L}(V)$." Let $\mathrm{SEP}^r(\mathscr{S})$ denote the set of right separators of $\mathscr{S}$ in $\mathscr{L}(V)$. Then, for X a Banach space, the analogues of Corollary 5 and Proposition 6 show that for finite dimensional subspaces $\mathscr{S} \subseteq \mathscr{B}(X)$, $\mathrm{SEP}^r(\mathscr{S})$ is always open and dense in $\mathscr{B}(X)$. Part (iii) of Proposition 9 implies that Lemma 7 and Theorem 8 remain valid with "separating vector" replaced with either "separator of rank $\leq n$" or "separating n-tuple."

Recall that a subspace $E \subseteq H$ is *semiinvariant* for an algebra $\mathscr{A} \subseteq \mathscr{B}(H)$ if $E = M \ominus N$ with $M, N \in \mathrm{Lat}\,\mathscr{A}$ and $N \subseteq M$. The property we use is that if E is semiinvariant, the compression map $A \mapsto PA|_E$ is a homomorphism, where $P = \mathrm{proj}(E)$.

Theorem 11. *Let H be a separable Hilbert space and let $\mathscr{A} \subseteq \mathscr{B}(H)$ be a weakly closed unital operator algebra for which there is an increasing sequence $E_n \subseteq E_{n+1}$ of semiinvariant subspaces with closed union H, such that each compression algebra $\mathscr{A}_n := P_n \mathscr{A}|_{E_n}$ has a separating vector and is finite dimensional, where $P_n = \mathrm{proj}(E_n)$. Then $\mathrm{sep}(\mathscr{A})$ is dense in H. So in particular, $\mathrm{sep}(\mathscr{A})$ is nonempty.*

Proof. Suppose $\mathrm{sep}(\mathscr{A})$ fails to be dense. Then there exists a closed ball J of positive radius with $J \cap \mathrm{sep}(\mathscr{A}) = \emptyset$. For each pair of positive integers n, m let

$$\mathscr{K}_{nm} := \{y \in J \colon \exists A \in \mathscr{A},\ Ay = 0,\ \|P_n A P_n\| = 1,\ \|A\| \leq m\}.$$

Then our assumption implies that $\bigcup_{n,m=1}^{\infty} \mathscr{K}_{nm} = J$. We first show that $\mathscr{K}_{nm}$ is closed. Suppose $y_k \in \mathscr{K}_{nm}$, $y_k \to y$. For each k let A_k be an operator in $\mathscr{A}$ such that $A_k y_k = 0$, $\|A_k\| \le m$, $\|P_n A_k P_n\| = 1$. By weak compactness and metrizability of the unit ball of $\mathscr{B}(H)$, there is a subsequence $\{A_{k_\ell}\}$ which converges weakly to an operator $A \in \mathscr{A}$. We have $\|A\| \le m$. For each n, the subsequence $P_n A_{k_\ell}|_{E_n}$ converges weakly to $P_n A|_{E_n}$, and since $\mathscr{A}_n$ is finite dimensional this convergence is in *norm*. So $\|P_n A P_n\| = 1$. We have $\|A_{k_\ell} y\| = \|A_{k_\ell}(y - y_{k_\ell})\| \le m\|y - y_{k_\ell}\|$ so $A_{k_\ell} y \to 0$. Hence $\|Ay\|^2 = \langle Ay, Ay\rangle = \mathrm{Lim}_\ell \langle A_{k_\ell} y, Ay\rangle = 0$. This shows that $y \in \mathscr{K}_{nm}$ as required.

Since the sets $\mathscr{K}_{nm}$ are closed and $\bigcup_{n,m} \mathscr{K}_{nm} = J$, by the Baire Category Theorem some $\mathscr{K}_{nm}$ must contain a ball $B_\varepsilon(y) := \{x \in H : \|x - y\| < \epsilon\}$. Density for $\bigcup_n P_n H$ implies that $B_\varepsilon(y) \cap E_q$ is not empty for some $q \ge n$. Since $\mathscr{A}_q$ has a separating vector and is a finite dimensional subalgebra of $\mathscr{B}(E_q)$, Corollary 5 implies that $\mathrm{sep}(\mathscr{A}_q)$ is dense in E_q. Hence $B_\varepsilon(y) \cap \mathrm{sep}(\mathscr{A}_q)$ is nonempty. Fix $z \in B_\varepsilon(y) \cap \mathrm{sep}(\mathscr{A}_q)$. We have $\mathscr{A}_q \ne \{0\}$ so $z \ne 0$. Since $z \in \mathscr{K}_{nm}$ there exists $A \in \mathscr{A}$, $Az = 0$, $\|P_n A P_n\| = 1$. Then $\|P_q A P_q\| \ge 1$. So $P_q A|_{E_q}$ is a nonzero element of $\mathscr{A}_q$ which annihilates z. This is our required contradiction. □

Theorem 11 yields an affirmative answer to Question 1 in [HLW].

Corollary 12. *Let H be a separable Hilbert space, and let $T \in \mathscr{B}(H)$ be a triangular operator. Then $\mathrm{sep}(\mathscr{W}(T))$ is dense in H. More generally, if $\mathrm{Lat}(T)$ contains a complete multiplicity one nest order isomorphic to a subset of the extended integers, then $\mathrm{sep}(\mathscr{W}(T))$ in dense in H.*

Proof. It is well known that a singly generated algebra acting on a finite dimensional space has a separating vector. So if T has a triangular basis $\{e_n\}$, then the invariant subspaces $E_n = [e_1, \dots, e_n]$ satisfy the hypotheses of Theorem 11 for $\mathscr{A} = \mathscr{W}(T)$. Similarly, if $\mathrm{Lat}(T)$ contains a nest of the prescribed type, then there is a sequence of finite dimensional interval projections for the nest which satisfy the hypotheses of Theorem 11 for $\mathscr{W}(T)$. □

Remark 13. As pointed out in [HLW], a Hilbert space operator T is triangular iff the set of *algebraic vectors* for T is dense in H, where a vector x is called algebraic for T if $p(T)x = 0$ for some nontrivial polynomial $p(z)$. This makes sense in a separable Banach space setting. Density of the algebraic vectors for $T \in \mathscr{B}(X)$ is equivalent to the existence of a sequence $\{x_n\}$ in X with dense span such that $E_n := [x_1, \dots, x_n]$ is invariant under T for each n, so this is a natural extension of the notion of triangularity. If X is reflexive Corollary 12 extends to this case. To adapt the proof of Theorem 11 let $\{P_n\}$ be a commuting sequence of bounded projections from X onto the finite dimensional spaces $\{E_n\}$. Although $\{P_n\}$ may not be uniformly bounded, this poses no obstruction, and the proof carries through with only minor modifications.

Questions. Our proofs suggest some new directions. Suppose $\operatorname{sep}(\mathscr{S}) \neq \emptyset$. If L is a complex line meeting $\operatorname{sep}(\mathscr{S})$, define

$$i(\mathscr{S};L) = \operatorname{card}\{y \in L\colon\ y \notin \operatorname{sep}(\mathscr{S})\},$$

and for $x \in \operatorname{sep}(\mathscr{S})$ define

$$j(\mathscr{S};x) = \sup\{(i(\mathscr{S};L)\colon L \text{ is a complex line containing } x\}.$$

Then define $k(\mathscr{S}) = \sup_x j(\mathscr{S};x)$. This gives a way of associating natural *indices of separation* to lines meeting $\operatorname{sep}(\mathscr{S})$, points in $\operatorname{sep}(\mathscr{S})$, and to subspaces $\mathscr{S}$. By Proposition 3, $k(\mathscr{S}) \leq \dim(\mathscr{S})$. It might be worthwhile to investigate this further. If T acts on a finite dimensional space, it is not hard to show that $k\big(\mathscr{W}(T)\big)$ is the cardinality of the spectrum of T. So the index $k(\mathscr{S})$ somehow generalizes this concept. For another direction, let us say that a nonempty set $\mathscr{E}$ of vectors is *linearly dense* if $\mathscr{E} \cap L$ is dense in L for all complex lines L that meet $\mathscr{E}$. This is a stronger property than density. Proposition 4 implies that if $\mathscr{S}$ is a linear subspace of operators with denumerable Hamel basis, then $\operatorname{sep}(\mathscr{S})$ is either empty or linearly dense. Also, it is not hard to show (see [LW2], Lemma 8) that if $\mathscr{S}$ is a m.a.s.a. on a separable Hilbert space H, then $\operatorname{sep}(\mathscr{S})$ is linearly dense in H. Is linear density common? More to the point, if T is triangular, is $\operatorname{sep}\big(\mathscr{W}(T)\big)$ linearly dense?

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